Exercise 36

Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$, if the density at any point is equal to the square of the distance from the origin.

Solution

The linear density of the wire is

$$\rho(x, y, z) = \left(\sqrt{x^2 + y^2 + z^2}\right)^2 = x^2 + y^2 + z^2.$$

Integrate this density over the length of the wire to obtain the total mass.

$$\begin{aligned} M = \int_C \rho\, ds &= \int_0^{2\pi} \{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} (t^2 + \cos^2 t + \sin^2 t)\sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2}\, dt \\ &= \int_0^{2\pi} (t^2 + 1)\sqrt{2}\, dt \\ &= \sqrt{2}\left(\frac{t^3}{3} + t\right)\bigg|_0^{2\pi} \\ &= \sqrt{2}\left(\frac{8\pi^3}{3} + 2\pi\right) \\ &= \frac{2\sqrt{2}\pi}{3}(4\pi^2 + 3) \end{aligned}$$

Calculate the x-coordinate of the center of mass.

$$\bar{x} = \frac{\displaystyle\int x\, dm}{\displaystyle\int dm} = \frac{\displaystyle\int_C x(\rho\, ds)}{M}$$

$$\begin{aligned} &= \frac{3}{2\sqrt{2}\pi(4\pi^2 + 3)} \int_0^{2\pi} x(t)\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \frac{3}{2\sqrt{2}\pi(4\pi^2 + 3)} \int_0^{2\pi} t(t^2 + \cos^2 t + \sin^2 t)\sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2}\, dt \\ &= \frac{3}{2\sqrt{2}\pi(4\pi^2 + 3)} \int_0^{2\pi} t(t^2 + 1)\sqrt{2}\, dt \\ &= \frac{3}{2\pi(4\pi^2 + 3)}(2\pi^2 + 4\pi^4) \\ &= \frac{3\pi(1 + 2\pi^2)}{4\pi^2 + 3} \approx 4.60 \end{aligned}$$

Calculate the y-coordinate of the center of mass.

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_C y(\rho\,ds)}{M}$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)} \int_0^{2\pi} y(t)\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)} \int_0^{2\pi} \cos t(t^2 + \cos^2 t + \sin^2 t)\sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2}\,dt$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)} \int_0^{2\pi} \cos t(t^2+1)\sqrt{2}\,dt$$

$$= \frac{3}{2\pi(4\pi^2+3)} \left(\int_0^{2\pi} t^2\cos t\,dt + \int_0^{2\pi} \cos t\,dt\right)$$

$$= \frac{3}{2\pi(4\pi^2+3)} \left[\int_0^{2\pi} t^2 \frac{d}{dt}(\sin t)\,dt + \int_0^{2\pi} \cos t\,dt\right]$$

$$= \frac{3}{2\pi(4\pi^2+3)} \left[\underbrace{(t^2\sin t)\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} \frac{d}{dt}(t^2)\sin t\,dt + \underbrace{(\sin t)\Big|_0^{2\pi}}_{=0}\right]$$

$$= \frac{3}{2\pi(4\pi^2+3)} \left[-\int_0^{2\pi} (2t)\sin t\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)} \left[\int_0^{2\pi} t\frac{d}{dt}(\cos t)\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)} \left[t\cos t\Big|_0^{2\pi} - \int_0^{2\pi} \frac{d}{dt}(t)\cos t\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)} \left[2\pi - \int_0^{2\pi} \cos t\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)} \left[2\pi - \underbrace{(\sin t)\Big|_0^{2\pi}}_{=0}\right]$$

$$= \frac{6}{4\pi^2+3}$$

$$\approx 0.141$$

Calculate the z-coordinate of the center of mass.

$$\bar{z} = \frac{\int z\,dm}{\int dm} = \frac{\int_C z(\rho\,ds)}{M}$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi} z(t)\{[x(t)]^2+[y(t)]^2+[z(t)]^2\}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi} \sin t(t^2+\cos^2 t+\sin^2 t)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\,dt$$

$$= \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi} \sin t(t^2+1)\sqrt{2}\,dt$$

$$= \frac{3}{2\pi(4\pi^2+3)}\left(\int_0^{2\pi} t^2\sin t\,dt+\int_0^{2\pi}\sin t\,dt\right)$$

$$= \frac{3}{2\pi(4\pi^2+3)}\left[\int_0^{2\pi} t^2\frac{d}{dt}(-\cos t)\,dt+\int_0^{2\pi}\sin t\,dt\right]$$

$$= \frac{3}{2\pi(4\pi^2+3)}\left[t^2(-\cos t)\Big|_0^{2\pi}-\int_0^{2\pi}\frac{d}{dt}(t^2)(-\cos t)\,dt+\underbrace{(-\cos t)\Big|_0^{2\pi}}_{=0}\right]$$

$$= \frac{3}{2\pi(4\pi^2+3)}\left[-4\pi^2-\int_0^{2\pi}(2t)(-\cos t)\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)}\left(-2\pi^2+\int_0^{2\pi} t\cos t\,dt\right)$$

$$= \frac{3}{\pi(4\pi^2+3)}\left[-2\pi^2+\int_0^{2\pi} t\frac{d}{dt}(\sin t)\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)}\left[-2\pi^2+\underbrace{t\sin t\Big|_0^{2\pi}}_{=0}-\int_0^{2\pi}\frac{d}{dt}(t)\sin t\,dt\right]$$

$$= \frac{3}{\pi(4\pi^2+3)}\left(-2\pi^2-\int_0^{2\pi}\sin t\,dt\right)$$

$$= \frac{3}{\pi(4\pi^2+3)}\left(-2\pi^2-\underbrace{(-\cos t)\Big|_0^{2\pi}}_{=0}\right)$$

$$= -\frac{6\pi}{4\pi^2+3}$$

$$\approx -0.444$$

Therefore, the center of mass of the wire is roughly $(4.60, 0.141, -0.444)$.